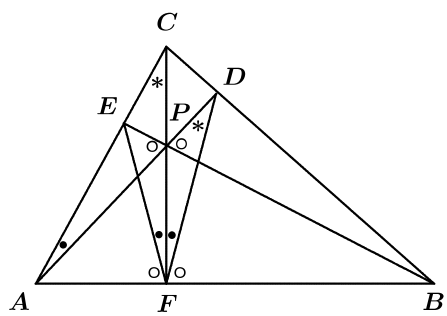
Now, if n is odd, then $n-k$ is even for each odd k and $\sum\limits_{k \text{ odd}} \binom{n}{k}(\sqrt{a})^{n-k}$ is an integer so that A is an integer multiple of 4.

If n is even, then $2\sum\limits_{k \text{ odd}} \binom{n}{k}(\sqrt{a})^{n-k} = 2(\sqrt{a})\cdot B$ for some integer B and $A = 4aB^2$ is an integer multiple of 4 as well.

6. Let ABC be a triangle. Let D, E, and F be points on the line segments BC, CA, and AB, respectively, such that line segments AD, BE, and CF meet in a single point. Suppose that $ACDF$ and $BCEF$ are cyclic quadrilaterals. Prove that AD is perpendicular to BC, BE is perpendicular to AC, and CF is perpendicular to AB.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.



Let P be the point at which AD, BE, and CF meet.

Since $ACDF$ is cyclic, $\angle ACF = \angle ADF$; since $BCEF$ is cyclic, $\angle ECF = \angle EBF$. Therefore, $PDBF$ is cyclic. Analogously, $PEAF$ is cyclic.

Now, $\angle EFA = \angle EPA = \angle DPB = \angle DFB$. Also, $\angle PFE = \angle PAE = \angle PFD$ (the latter equality holds since $ACDF$ is cyclic). Thus, $\angle CFA = \angle CFB = 90^\circ$. It follows that $\angle BEA$ and $\angle ADB$ are each 90°.

7. Let a_0, a_1, a_2, $\ldots$ and b_0, b_1, b_2, $\ldots$ be two sequences of integers such that $a_0 = b_0 = 1$ and for each nonnegative integer k

(a) $a_{k+1} = b_0 + b_1 + b_2 + \cdots + b_k$, and

(b) $b_{k+1} = (0^2+0+1)a_0 + \big(1^2+1+1\big)a_1 + \cdots + \big(k^2+k+1\big)a_k$.

For each positive integer n show that

$$a_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n}.$$

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We use Alt's solution.

The recursions (a) and (b) can be rewritten as follows:

$$\begin{aligned} a_{n+1} &= a_n + b_n\,, \\ b_{n+1} &= \big(n^2+n+1\big)\,a_n + b_n\,; \quad n \ge 1\,. \end{aligned} \tag{1}$$

By making the substitutions $b_n = a_{n+1} - a_n$ and $b_{n+1} = a_{n+2} - a_{n+1}$ in $b_{n+1} = \left(n^2 + n + 1\right) a_n + b_n$ we obtain successively

$$\begin{aligned} a_{n+2} - a_{n+1} &= \left(n^2 + n + 1\right) a_n + a_{n+1} - a_n \,, \\ a_{n+2} &= 2a_{n+1} + n\,(n+1)\,a_n \,, \\ a_{n+2} &= 2a_{n+1} + n\,(n+1)\,a_n \,; \qquad n \geq 1 \,, \end{aligned} \tag{2}$$

where $a_0 = 1$ and $a_1 = b_0 = 1$.

Using (2) we get $a_2 = 2$, $a_3 = 6$, $a_4 = 24$, and $a_5 = 120$, suggesting that $a_n = n!$, and we confirm this by using Mathematical Induction.

Indeed, supposing that $a_n = n!$ and $a_{n-1} = (n-1)!$ and using (2) we obtain, for any $n \geq 1$,

$$\begin{aligned} a_{n+1} &= 2a_n + (n-1)na_{n-1} &&= 2n! + (n-1)n(n-1)! \\ &= 2n! + (n-1)n! &&= (n-1+2)n! \;=\; (n+1)! \,. \end{aligned}$$

Since $a_n = n!$, then $b_n = a_{n+1} - a_n = (n+1)! - n! = n \cdot n! = na_n$, and therefore

$$\frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} = \frac{n! a_1 a_2 \cdots a_n}{a_1 a_2 \cdots a_n} = n! = a_n \,.$$

Next we look at solutions from our readers to problems of the 56th Belarusian Mathematical Olympiad 2006, Category C, Final Round, given at [2009 : 147–148].

1. (E. Barabanov) Is it possible to partition the set of all integers into three nonempty pairwise disjoint subsets so that for any two numbers a and b from different subsets

(a) there is a number c in the third subset such that $a + b = 2c$?

(b) there are numbers c_1 and c_2 in the third subset such that $a + b = c_1 + c_2$?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

(a) This is impossible. Suppose $\mathbb{Z} = A \cup B \cup C$ is a partition of $\mathbb{Z}$ satisfying the given condition. Without loss of generosity, assume $1 \in A$. If B contains any even integer b, then $1 + b$ is odd. Since $2c$ is even for all $c \in C$, we have a contradiction. Hence, B contains no even integers. Then $2 \in A$ or $2 \in C$. In either case, $2 + b$ is odd for any $b \in B$, again a contradiction.

(b) This is possible. Let $\mathbb{Z}$ be partitioned as $\mathbb{Z} = U \cup V \cup W$ where $U = \{3k \mid k \in \mathbb{Z}\}$, $V = \{3k+1 \mid k \in \mathbb{Z}\}$, and $W = \{3k+2 \mid k \in \mathbb{Z}\}$. Let a and b be two numbers from different subsets in the partition. There are three cases to consider:

If $a \in U$, $b \in V$, then write $a = 3k_1$ and $b = 3k_2 + 1$, and take $c_1 = 3k_1 + 2$ and $c_2 = 3(k_2 - 1) + 2$ as the required elements in W.